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1989 J. Phys. A: Math. Gen. 22 L93

(http://iopscience.iop.org/0305-4470/22/3/006)

J. Phys. A: Math. Gen. 22 (1989) L93-L97. Printed in the UK

LETTER TO THE EDITOR

The trail problem on finitely ramified fractals

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Received 10 October 1988

Abstract. Using an exact real-space renormalisation group technique, we show that self-avoiding trails (SAT) and self-avoiding walks (SAW) on the Sierpinski gasket obey the same critical exponent ν for the 'correlation length' and therefore belong to the same universality class. On the other hand, it is shown that SAT on branching Koch curves with maximum ramification number $R_{max} > 3$ is in a different universality class from that of SAW.

The self-avoiding walk (SAW) has long been successfully used to model the 'excluded volume' effect of a linear polymer in dilute solutions (Domb 1963). It has also been related to other lattice problems, such as the Ising model of ferromagnetism and the n-component spin model (Domb 1970, McKenzie 1967, de Gennes 1972). Stimulated by these successes, several authors have proposed various generalised versions of the SAW, also with 'excluded volume' effects, to describe the different kinds of statistical and kinetic behaviour of linear polymers, e.g. the k-tolerant walk model for the description of real polymers and growing SAW for θ-point behaviour (Malakis 1976, Turban 1983, Majid *et al* 1984, Kremer and Lyklema 1985, Lyklema 1985, Turban and Debierre 1987). Among these generalisations of SAW, the self-avoiding trails (SAT) is the simplest one used to consider the effect of loop formation on the statistical properties of polymers (Malakis 1976).

As a kind of generalised equilibrium SAW, the SAT is constructed similarly to the SAW except that each 'bond' rather than each 'site' can be passed only once. In an earlier paper, Malakis (1976) studied the SAT on the square lattice and suggested that certain critical exponents have the same values for both SAW and SAT and therefore they are in the same universality class. In contrast, using a one-parameter $b \times b$ cell-to-bond real-space renormalisation group (RSRG) transformation, Li *et al* (1984) questioned this and suggested that the value of the 'end-to-end distance' critical exponent for trails may be located between the values corresponding to ordinary random walk ($\nu = 0.5$) and to SAW ($\nu = 0.75$). In addition, Zhou and Li (1984) carefully analysed this trail problem by the series expansion method and pointed out that one may expect the possibility of the emergence of a new universality class for SAT. In order to investigate this problem further, Malakis (1984) reconsidered the trail problem using a two-parameter cell-to-bond RSRG technique and conjectured that SAT and SAW on the square lattice are in the same universality class. Owing to the approximate methods used by these authors, there is no conclusive evidence as to whether SAT and SAW are in the same universality class. Fortunately, the self-similarity of deterministic fractals, such as the Sierpinski gasket (SG) (Gefen *et al* 1984) and Koch curves (Gefen *et al* 1983), offers the possibility of studying this problem by the exact RSRG technique. In this letter, we consider the SAT on the SG and the branching Koch curves with maximum ramification number $R_{max} > 3$ using an exact RSRG technique. The critical exponent ν obtained on the SG is found to be just the same as that found by Ben-Avraham and Havlin (1984) for the SAW. On the branching Koch curves with $R_{max} > 3$, however, it can be shown that the critical exponents for the SAT and the SAW obey different values.

Following the RSRG technique presented by Ben-Avraham and Havlin (1984) in studying SAW, we consider SAT on the SG shown in figure 1. For the SAT problem, we need nine quantities similar to those of Ben-Avraham and Havlin (1984) for the SAW. They are defined as follows.

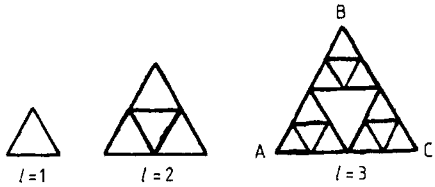

Figure 1. Sierpinski Gaskets of order $l = 1, 2$ and 3.

$P'_1 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at B without passing through C.

$P'_2 \equiv$ the number of SAT in which the walker enters the gasket at A, passes through C and then exits at B.

$P'_3 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at A, without visiting either B or C.

$P'_4 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at A, having visited either B or C. (Due to the equilibrium property of the SAT, P'_4 may also be equivalently defined as the number of SAT in which the walker enters the gasket at A and exits at B and then re-enters the same gasket at B and exits at A, without visiting C either time. Similarly, the following quantities P'_5–P'_9 may have more than one equivalent definition.)

$P'_5 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at A, having visited both B and C.

$P'_6 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at A, having visited neither B nor C, and then re-enters the same gasket at B (C) and exits at C (B).

$P'_7 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at A, having visited neither B nor C, and then re-enters the same gasket at B (C) and exits at B (C) without visiting C (B).

$P'_8 \equiv$ the number of SAT in which the walker enters the gasket at A and exits at A, having visited neither B nor C, and then re-enters the same gasket at B (C) and exits at B (C), having visited C (B).

$P'_9 \equiv$ the number of SAT in which the walker enters and then exits the gasket at A, B, C successively in the way similar to that of P_3.

Inserting the initial values $P_1 = P_2 = 1$, $P_5 = 2$, $P_3 = P_4 = P_6 = P_7 = P_8 = P_9 = 0$ to the recursion relations for $\{P_i\}$ (which are so complicated that we have omitted them from this letter for simplicity) then after iteration we have the following relations for $l \gg 1$:

$$\begin{array}{lll} P_7' = P_8' & 4P_3' = 2P_4' = P_5' & P_1'/P_5' \to 0 \\ P_2'/P_5' \to 0 & P_6'/P_5' \to 0 & P_5'/P_7' \to 0 \\ P_5'P_9'/(P_8')^2 = \mu & P_5'P_6'/P_2'P_7' = \omega & \end{array} \tag{1}$$

where μ and ω are finite with their fixed points determined by the equations

$$\mu^* = (84 + 72\mu^*)/(7 + \mu^*)^2 \tag{2a}$$

$$\omega^* = 3(\mu^* + 4\omega^*)/(7 + \mu^*)(1 + \omega^*). \tag{2b}$$

With the use of relations (1), the recursion relations for the $\{P_i\}$ reduce to (valid only for $l \gg 1$):

$$P_1' = 2P_2P_4P_6 + P_2^2P_7 \tag{3a}$$

$$P_2' = 2P_2P_5P_6 + 2P_2^2P_8 \tag{3b}$$

$$P_3' = 2P_4P_5P_7 + 2P_4^2P_8 \tag{3c}$$

$$P_4' = 4P_4P_5P_6 + P_5^2P_7 \tag{3d}$$

$$P_5' = 6P_5^2P_8 \tag{3e}$$

$$P_6' = 4P_2P_6P_8 + P_2^2P_9 \tag{3f}$$

$$P_7' = 4P_5P_7P_8 + 2P_4P_5P_9 + 6P_4P_8^2 \tag{3g}$$

$$P_8' = 7P_5P_8^2 + P_5^2P_9 \tag{3h}$$

$$P_9' = 14P_8^3 + 12P_5P_8P_9. \tag{3i}$$

Equations (1), (3a) and (3b) yield

$$P_1' = P_2'/2. \tag{4}$$

Let N_1 be the mean number of steps for a SAT in the manner of P_1, i.e. N_1 is the mean number of steps for a SAT in which a walker enters the gasket at A and exits at B without visiting C. Similarly, one can define N_i ($i = 2, \ldots, 9$) as the mean number of steps for a SAT in the manner of P_i. In order to find the critical exponent ν using a technique similar to that of Ben-Avraham and Havlin (1984), we need only consider the recursion relations (3a), (3e) and (3g), which present a closed form

$$P_1' = 4(1+\omega)P_1^2P_7 \qquad P_5' = 6P_5^2P_7 \qquad P_7' = (7+\mu)P_5P_7^2 \tag{5}$$

where (1) and (4) have been used. Weighting the contribution of N_k by P_k ($k = 1, 5, 7$), we have the recursion relations for N_k

$$\begin{aligned} P_1'N_1' &= 8(1+\omega)P_1^2P_7N_1 + 4(1+\omega)P_1^2P_7N_7 \\ P_5'N_5' &= 12P_5^2P_7N_5 + 6P_5^2P_7N_7 \\ P_7'N_7' &= (7+\mu)P_5P_7^2N_5 + 2(7+\mu)P_5P_7^2N_7. \end{aligned} \tag{6}$$

Noting the closed form of the recursion relations (5) for $l \gg 1$, one gets

$$\begin{pmatrix} N_1' \\ N_5' \\ N_7' \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_5 \\ N_7 \end{pmatrix}. \tag{7}$$

The transforming matrix has the maximum eigenvalue $\lambda_{max}=3$. Thus for SAT on the SG one has

$$\nu = \ln b/\ln \lambda_{max} = \ln 2/\ln 3 \tag{8}$$

which is just the same value obtained by Ben-Avraham and Havlin (1984) for SAW on the SG. Thus one can conclude that SAT and SAW on the SG are in the same universality class.

An important parameter of a Koch curve is the order of ramification R. At a point P, R measures the smallest number of significant interactions which one must cut in order to isolate an arbitrary bounded set of points connected to P (Gefen *et al* 1983). It is easy to show that on Koch curves with maximum ramification number $R_{max} \leqslant 3$, such as shown in figure 2(*a*), SAT and SAW are just the same. On Koch curves with $R_{max}>3$, one may expect an essential difference between SAT and SAW. As an example, we study the SAT on the Koch curve with $R_{max}=4$ as shown in figure 2(*b*). Let P_1' be the total number of SAT which can be performed from A to B, without passing through either C or D, P_2' the number of SAT which can be performed from A to B, passing through either C or D, and P_3' the number of SAT which can be performed from A to B, passing through both C and D. Obviously, one has the recursion relations

$$\begin{aligned} P_1' &= (P_1+2P_2+6P_3)^3 \\ P_2' &= (P_1+2P_2+6P_3)^4 \\ P_3' &= (P_1+2P_2+6P_3)^7. \end{aligned} \tag{9}$$

It is clear that P_3' is dominant, i.e. when l is large enough, one has $P_1'/P_3' \to 0$ and $P_2'/P_3' \to 0$. Thus (9) reduce to $P_3'=(6P_3)^7$, yielding

$$\nu = \ln 3/\ln 7. \tag{10}$$

This value is rather different from the corresponding value $\nu=\ln 3/\ln 4$ easily obtained for SAW on the same Koch curve. We suggest that SAT on all branching Koch curves with $R_{max}>3$ are in different universality classes from those of SAW.

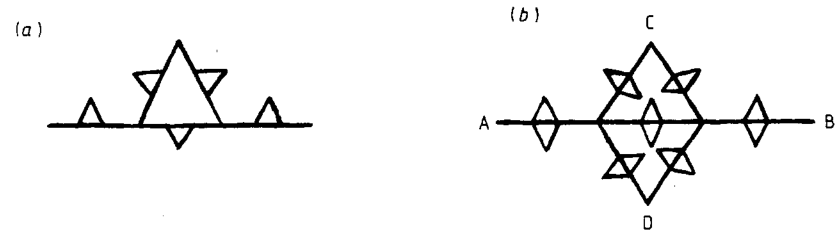

Figure 2. Koch curves with (*a*) $R_{max}=3$ and (*b*) $R_{max}=4$.

In conclusion, we have considered the trail problem on some finitely ramified fractals using an exact RSRG technique. We have shown that SAT and SAW on the SG possess the same critical exponent ν and therefore belong to the same universality class. This seems to show the possibility that SAT and SAW on the square lattice are in the same universality class. On the other hand, we showed that SAT and SAW on the Koch curves with $R_{max}>3$ are in different universality classes. These results lend themselves to our conclusion that whether or not SAT and SAW on fractal lattices are in the same universality class depends on the underlying structure.

This work has been supported by the Foundation of Doctor Education and National Science Foundation M No 85127.

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